Naval Research Laboratory
Stennis Space Center, MS 39529-5004

NRL/MR/7441-93-7079

fBm Interpolator

ANDREW B. MARTINEZ

Department of Electrical Engineering
Tulane University

Prepared for Marine Geosciences Division

November 4, 1994

REPORT DOCUMENTATION PAGE

Form Approved
OBM No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
November 4, 1994

3. REPORT TYPE AND DATES COVERED
Final

4. TITLE AND SUBTITLE
fBm Interpolator

5. FUNDING NUMBERS
Job Order No. 574513105
Program Element No. 0602435N
Project No. R035
Task No. S41
Accession No. DN252201

6. AUTHOR(S)
Andrew B. Martinez*

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Department of Electrical Engineering
Tulane University
New Orleans, LA 70118

8. PERFORMING ORGANIZATION REPORT NUMBER
NRL/MR/7441--93-7079

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Office of Naval Research
800 N. Quincy Street
Arlington, VA 22217-5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES
*Prepared for Marine Geosciences Division, Stennis Space Center, MS 39529-5004

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
A technique for interpolating an irregularly sampled multidimensional field is presented. The method assumes that the field to be interpolated is a fractional Brownian motion surface with additive measurement error. There are several advantages to this method: data sets with different mean-squared measurement errors can be optimally merged, and the variance of the estimator can be easily computed at each interpolation point.

An iterative algorithm is given for implementing the estimator with the advantage of allowing a stopping rule to be used to terminate the iteration after a predetermined level of performance is achieved. Furthermore, at each stage, multiple points can be tested, the resulting performance evaluated, and that point which most decreases the variance of the estimator selected. Using the iterative algorithm produces a low order estimator based on a small, carefully selected set of points with a reasonable level of performance.

14. SUBJECT TERMS
geophysical algorithms, fractional Brownian motion, interpolation

15. NUMBER OF PAGES
15

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	SAR

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. Z39-18
298-102

CONTENTS

fBm Interpolator

Andrew B. Martinez
Department of Electrical Engineering
Tulane University
New Orleans, LA 70118
(504)865-5785

October 31, 1994

1.0 Introduction

In fitting data, some assumptions about the process to be modelled are inevitably made, either implicitly or explicitly. There is much empirical evidence that many naturally occurring processes are modelled well by fractional Brownian Motion (fBm), and this model can form the basis for a model-based statistical estimator, the fBm interpolator.

While interpolation is often considered for the case of doubles (x, z) or triples (x, y, z), the derivation that follows is easily generalized to data of higher dimension. For this reason, it is convenient to represent position as a vector (e.g., $\vec{\alpha} = (x, y)$).

Given n data points of the form $(\vec{\alpha}_i, z_i)$, the interpolation problem can be considered as an estimation problem: find the best estimate of z at the location $\vec{\alpha}$. Obviously, finding the best estimator using classical methods requires a detailed knowledge of the statistics of the process.

2.0 Statistical Model of the Data

2.1 Error-Free Data

Let us assume $(\vec{\alpha}_i, z_i)$, $i = 1, \ldots, n$ to be n error-free samples of a fractional Brownian motion. While we assume no error in the data (i.e., the data values are exact), this assumption is probably not realistic, and it is relaxed in section 2.2.

If $(\vec{\alpha}_i, z_i)$ is fBm, then it is well known that the increments $z_i - z_j$ are Gaussian with zero mean and variance proportional to the spacing of the points:

$$\mathrm{var}(z_i - z_j) = \sigma^2 \|\vec{\alpha}_i - \vec{\alpha}_j\|^{2H} \tag{1}$$

where $\|\cdot\|$ represents the usual Euclidean norm. We assume the statistics to be locally homogeneous, that is H and σ^2 are assumed locally constant. The exponent H describes the roughness of the surface, and it must be estimated. It takes values in the range $0 < H < 1$, where $H = 0.5$ corresponds to true Brownian motion, and empirical evidence indicates that $0.7 < H < 0.99$ corresponds to many natural phenomena. The scale factor σ^2 can be interpreted as the mean-squared "variability" of the fBm. More detail on the mathematical foundations of fBm can be found in [1].

To simplify the notation that follows, we define the quantities

$$\rho_i = \|\vec{\alpha}_i - \vec{\alpha}\|^{2H} \quad \text{and} \quad \rho_{i,j} = \|\vec{\alpha}_i - \vec{\alpha}_j\|^{2H}. \tag{2}$$

Thus we can write $\mathrm{var}(z_i - z) = \sigma^2 \rho_i$ and $\mathrm{var}(z_i - z_j) = \sigma^2 \rho_{i,j}$. Furthermore, since z is fBm,

$$\mathrm{cov}(z_i, z_j) = (\sigma^2/2)(\rho_i + \rho_j - \rho_{i,j}). \tag{3}$$

Let $\mathbf{z}$ be a vector of data values, $\mathbf{z} = [z_i]$ for $i = 1, \ldots, n$. Since $\mathbf{z}$ and the interpolation point $(\vec{\alpha}, z)$ are sampled fBm, the data vector $\mathbf{z} - z\mathbf{1}$ (where $\mathbf{1}$ is a vector of all ones) is multivariate Gaussian with zero mean and covariance $\mathbf{\Sigma}$. Viewed another way, $\mathbf{z}$ is multivariate Gaussian with mean $z\mathbf{1}$ and covariance $\mathbf{\Sigma}$.

Assuming the unknown z value is fixed, the covariance matrix is given by

$$\begin{aligned} \mathbf{\Sigma} &= \mathrm{E}\mathbf{z}\mathbf{z}' - z^2\mathbf{1}\mathbf{1}' \\ &= \sigma^2\mathbf{R} \end{aligned} \tag{4}$$

where the matrix $\mathbf{R}$ has elements $r_{i,j} = (\rho_i + \rho_j - \rho_{i,j})/2$. Note that the covariance is factored into $\mathbf{R}$ a function of data and interpolant positions, and σ^2. In the error-free case, we can generate an estimator that is independent of σ^2.

2.2 Data with Additive Errors

Assume the data $\mathbf{z}$ to be a fBm vector $\tilde{\mathbf{z}}$ with additive Gaussian error $\mathbf{e}$, that is $\mathbf{z} = \tilde{\mathbf{z}} + \mathbf{e}$. The covariance of $\mathbf{z}$ can be easily found. The statistics of the fBm vector $\tilde{\mathbf{z}}$ are given in Equation(4) for the error-free case. Let the error

have zero mean and covariance $\mathrm{E}\mathbf{e}\mathbf{e}' = \eta^2\mathbf{C}$ and independent of the fBm, $\tilde{\mathbf{z}}$. It follows that the covariance matrix of $\mathbf{z}$ is given by

$$\boldsymbol{\Sigma} = \sigma^2\mathbf{R} + \eta^2\mathbf{C} \tag{5}$$

where $\mathbf{R}$ is defined in Equation (4), and $\mathbf{C}$ describes the mean-squared error in the data.

In addition to σ^2, this introduces a new quantity, the error covariance, $\eta^2\mathbf{C}$. Unfortunately, unlike the error-free case, we cannot generate an estimator that is independent of η^2, σ^2, and $\mathbf{C}$. Fortunately, it is frequently possible to determine the covariance structure.

Several cases come immediately to mind. In the simplest case, we assume iid measurement errors with zero mean and variance η^2, then we can write $\mathbf{C} = \mathbf{I}$, and $\boldsymbol{\Sigma} = \sigma^2\mathbf{R} + \eta^2\mathbf{I}$. The resulting estimator requires one additional parameter not needed in the error–free case: η^2/σ^2.

While this may be true of some data sets, frequently, as in the case of multibeam sonar bathymetry, there are systematic and unavoidable errors in the data collection that produce different levels of error at different points. In this case, $\mathbf{C}$ might be assumed to be diagonal (independent errors at adjacent points), but not constant on the diagonal. Merging data sets with different mean-squared errors will produce the same result. Finally, in the most general case $\mathbf{C}$ need not be diagonal.

3.0 The Maximum Likelihood Estimator

Viewed from the position $\vec{\alpha}$, fBm can be expressed as a Gaussian vector with constant mean $z\mathbf{1}$ and covariance $\boldsymbol{\Sigma}$. Thus the estimation of z at $\vec{\alpha}$ can be restated as the estimation of the mean of a Gaussian vector. This is a well known problem in statistics and the Maximum Likelihood Estimator is presented here without proof. For further information see Mardia et al.[2].

Given a multivariate Gaussian with mean $z\mathbf{1}$ and covariance $\boldsymbol{\Sigma}$, the maximum likelihood estimate of z is easily found to be [2]

$$\hat{z} = \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{z}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} = \mathbf{w}'\mathbf{z}. \tag{6}$$

The estimator $\hat{z}$ is an unbiased minimum-variance estimator with expected value z and variance

$$\mathrm{var}(\hat{z})/\sigma^2 = 1/\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}. \tag{7}$$

Note that the weight vector $\mathbf{w}$ is a function of data positions $\vec{\alpha}_i$ and not of z-values.

3.1 Error-Free Data

For the special case of error-free data, the covariance of the data vector z was shown above to be $\mathbf{\Sigma} = \sigma^2\mathbf{R}$. Thus, the maximum likelihood estimator of Equation (6) becomes

$$\hat{z} = \frac{\mathbf{1}'\mathbf{R}^{-1}\mathbf{z}}{\mathbf{1}'\mathbf{R}^{-1}\mathbf{1}} = \mathbf{w}'\mathbf{z} \tag{8}$$

As promised, the estimator does not require knowledge of σ^2. The normalized variance ξ of the estimator is

$$\xi = \mathrm{var}(\hat{z})/\sigma^2 = 1/\mathbf{1}'\mathbf{R}^{-1}\mathbf{1} \tag{9}$$

The normalized variance will prove to be a useful measure of performance that is independent of σ^2.

3.2 Data with Additive Errors

For data with additive measurement error, the maximum likelihood estimate of z in Equation (6) becomes

$$\hat{z} = \frac{\mathbf{1}'(\mathbf{R} + \nu^2\mathbf{C})^{-1}\mathbf{z}}{\mathbf{1}'(\mathbf{R} + \nu^2\mathbf{C})^{-1}\mathbf{1}} = \mathbf{w}'\mathbf{z} \tag{10}$$

where $\nu^2 = \eta^2/\sigma^2$. The estimator is unbiased with expected value z and minimum normalized variance

$$\xi = \mathrm{var}(\hat{z})/\sigma^2 = 1/\mathbf{1}'(\mathbf{R} + \nu^2\mathbf{C})^{-1}\mathbf{1}. \tag{11}$$

Note that the weight vector $\mathbf{w}$ is a function of data positions $\vec{\alpha}_i$ the ratio η^2/σ^2, and the error correlation $\mathbf{C}$, but not of z-values.

4.0 Iterative Estimation of z

The estimator given in (6) can be implemented iteratively. This follows immediately from the observation that the n-point estimator $\hat{z}_{(n)}$ and the $(n+1)$-st data point are jointly Gaussian. While this may not reduce the computational burden of this routine given the high efficiency of the Cholesky decomposition that can be used in the direct solution of Equations (8) or (10), there are two potential benefits of this technique. Since the variance of the estimator is computed at each stage in the algorithm, a stopping rule can be used to terminate the iteration after a predetermined level of performance is achieved. At each

stage, multiple points can be tested, the resulting performance evaluated, and that point which most decreases the variance of the estimator can be selected. Using the iterative algorithm combined with these rules produces a low order estimator based on a small, carefully selected set of points with a reasonable level of performance.

4.1 Derivation of the Iterative Algorithm

Let $\hat{z}_{(n)} = \mathbf{z}'_{(n)}\mathbf{w}_{(n)}$ be the estimate of z based on the length n data vector $\mathbf{z}_{(n)}$. As stated above, $\hat{z}_{(n)}$ it is Gaussian with mean z and variance $\sigma^2\xi_{(n)}$. An updated estimate of z can be found using $\hat{z}_{(n)}$ and an additional data point $(\vec{\alpha}_{n+1}, z_{n+1})$. A new data vector can be made from the n-th order estimate and a new data point; this yields the vector $[\hat{z}_{(n)} \quad z_{n+1}]'$ with mean $[z \quad z]'$ and covariance

$$\mathbf{\Sigma}_{(n+1)} = \sigma^2 \begin{bmatrix} \xi_{(n)} & \mathbf{r}'_{(n)}\mathbf{w}_{(n)} \\ \mathbf{r}'_{(n)}\mathbf{w}_{(n)} & \rho_{n+1} \end{bmatrix} \tag{12}$$

where

$$\mathbf{r}_{(n)} = cov(z_{n+1}, \mathbf{z}_{(n)}) = 0.5[\rho_i + \rho_{n+1} - \rho_{i,n+1}] \quad i = 1, \ldots, n. \tag{13}$$

As above, the MLE of z is given by

$$\hat{z}_{(n+1)} = \frac{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{z}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}} = \frac{1}{\gamma}\begin{bmatrix} \rho_{n+1} - \mathbf{r}'_{(n)}\mathbf{w}_{(n)} \\ \xi_{(n)} - \mathbf{r}'_{(n)}\mathbf{w}_{(n)} \end{bmatrix}' \begin{bmatrix} \hat{z}_{(n)} \\ z_{n+1} \end{bmatrix} \tag{14}$$

where

$$\gamma = \xi_{(n)} + \rho_{n+1} - 2\mathbf{r}'_{(n)}\mathbf{w}_{(n)}. \tag{15}$$

The new normalized variance is

$$\xi_{(n+1)} = \det(\mathbf{R})/\gamma \tag{16}$$

where

$$\det(\mathbf{R}) = \xi_{(n)}\rho_{n+1} - (\mathbf{r}'_{(n)}\mathbf{w}_{(n)})^2. \tag{17}$$

Finally, the weights can be updated as follows:

$$\mathbf{w}_{(n+1)} = \frac{1}{\gamma}\begin{bmatrix} (\rho_{n+1} - \mathbf{r}'_{(n)}\mathbf{w}_{(n)})\mathbf{w}_{(n)} \\ \xi_{(n)} - \mathbf{r}'_{(n)}\mathbf{w}_{(n)} \end{bmatrix}. \tag{18}$$

4.2 The Iterative Algorithm

The iterative algorithm is presented below.

0. Assume the scalars $\xi_{(n)}$ and $\hat{z}_{(n)}$, and the vectors $\mathbf{w}_{(n)}$ and ρ_i for $i = 1, \ldots, n$ are known.
1. Get the new data point $(\vec{\alpha}_{n+1}, z_{n+1})$.
2. Compute ρ_{n+1} and $\rho_{i,n+1}$ for $i = 1, \ldots, n$ using Equation (2).
3. Compute $\mathbf{r}_{(n)}$ using Equation (13).
4. Compute $\xi_{(n+1)}$ using Equation (16).
5. Compute $\hat{z}_{(n+1)}$ using Equation (14).
6. Update the weights $\mathbf{w}_{(n+1)}$ using Equation (18).
7. Go to 1.

5.0 Example

As a simple example of the character of the fBm interpolator, consider a simple one dimensional interpolation with data points $(x, z) = \{(0.2, 8.0), (1.6, 6.0), (2.9, 8.0), (4.2, 11.0), (5.7, 13.0), (7.0, 13.5), (8.1, 14.5), (9.6, 15.0)\}$.

The results of interpolating this data are shown in Figures 1-5. The 8 data points are interpolated on the grid $x = 0 : 0.1 : 10$ using different values for H and ν^2. The data points are indicated in each plot by the asterisks.

In Figures 1 and 2, the data is assumed error-free, thus the interpolating function passes through the data points. Note however the markedly different behavior of the interpolating function as H is varied. For $H = 0.3$ in Figure 1, the function is "peaky," with cusps at the data points, returning rapidly to a mean value, and containing significant high frequency components. This corresponds to interpolating a surface with a high fractal dimension (and high dimension results in what we instinctively call a noisier surface). Most naturally occurring surfaces or functions tend to have much lower fractal dimension: $H = 0.9$ is a more typical value. In Figure 2, $H = 0.9$, and the result is a smoother function with dominant low frequency components.

Contrasting Figures 2 and 3 shows the effect of assuming additive measurement error. In Figure 2, the data is assumed error-free, in Figure 3 it is assumed to have variance 1.0. Note that the interpolating function in Figure 3

does not pass directly through the data points, and it has significantly less curvature. Furthermore, as ν^2 is increased, the interpolating function approaches a straight line.

In Figure 4, ν^2 is varied from point to point. The point (7.0,13.5) is assigned a variance 10 times that of the other points. As a result, the data point in not as tightly fit as the other points.

In Figure 5, the *normalized* variance of the estimator $\mathbf{var}\{\hat{z}\}/\nu^2$ is plotted versus the x values for the interpolation shown in Figure 4. As expected, the performance decreases as the interpolation point moves farther from the data values. Note that the variance never goes to zero due to the measurement error, and that it increases at both ends of the data. It is interesting to note the variance in the vicinity of the "bad" point $(7.0, 13.5)$. While the variance does increase in this region, it is still relatively small due to the contibutions of "better" neighboring points.

Several conclusions can be drawn from this example. First, the exponent H must be estimated, assumed, or derived for a reasonable fit to the data. Techniques for estimating H or equivalently fractal dimension are still an open question. The results of a fit using a nonzero measurement error will very not only reduce the mean-squared error of the estimator, but because it tends to "straighten" the interpolating function, it should reduce the severity of artifacts resulting from exact fits. Finally, through the use of different values of ν^2 at each point, the fit can be selectively "relaxed" at suspected or known bad points.

6.0 Summary

The maximum likelihood interpolator for fBm is presented above for data satisfying this well-defined statistical model. While this technique has its own set of problems, they are quite different from those of many interpolators currently in use.

Setting the measurement error equal to zero results in an interpolating function that passes through all data points, but the character of the fitting function can be modified by selection of the exponent H. This leads to an underlying problem, the correct selection of H. Presumably, this selection or estimation problem can best be understood in the context of a particular fBm process, and techniques for estimating H must be developed in that context. As an aside, choice of H affects the spectral character of the resulting interpolating function: small H results in a "high-pass" fit, large H in a "low-pass" fit.

Assuming nonzero additive error relaxes the fit as expected. This also has the effect of reducing the curvature in the fit. Assigning different mean-squared error values to data points allows data of different precision to be merged, and selectively relaxes the fit at poor data points. The value of mean-squared error at the data points is obviously data specific, and presumably it is known or can be approximated. In absence of this knowledge, a global variance can be used.

Although the iterative estimator derived above may not may not reduce the degree of computation required, it has two significant benefits. First, a stopping rule can be used to terminate the iteration after a predetermined level of performance is achieved by using the variance of the estimator computed at each stage in the algorithm. Second, the data point which most reduces the variance of the estimator can be selected at each stage of the iteration.

7.0 Acknowledgments

This work was supported by Mine Counter Measures (MCM) Coastal Sensing, program element PE 0602435N, project R035S41, project manager, Herbert C. Eppert, Jr. The author would like to thank Maria Kalcic for her advice and assistance in preparing this report.

References

[1] Kenneth Falconer, *"Fractal Geometry: Mathematical Foundations and Applications"*, John Wiley & Sons Ltd., Chichester, 1989.

[2] K. V. Mardia, J. T. Kent, and J. M. Bibby, *"Multivariate Analysis"*, Academic Press, London, 1979.

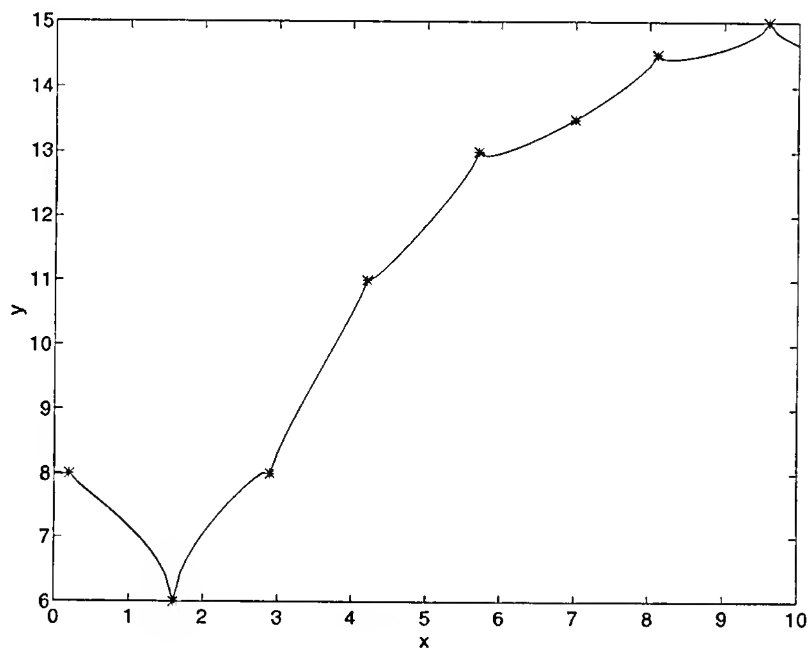

Figure 1: fBm interpolation ($H = 0.3$, $\nu^2 = 0.0$)

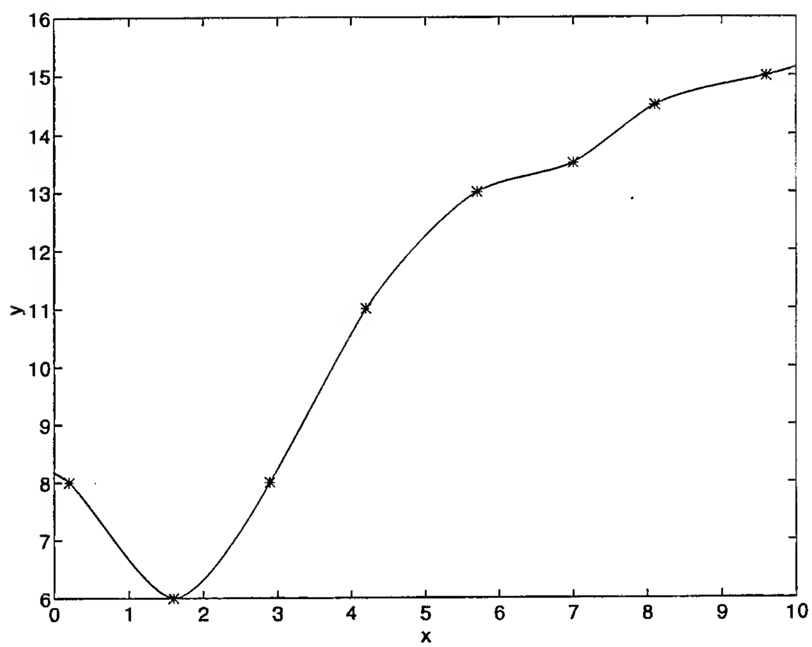

Figure 2: fBm interpolation ($H = 0.9$, $\nu^2 = 0.0$)

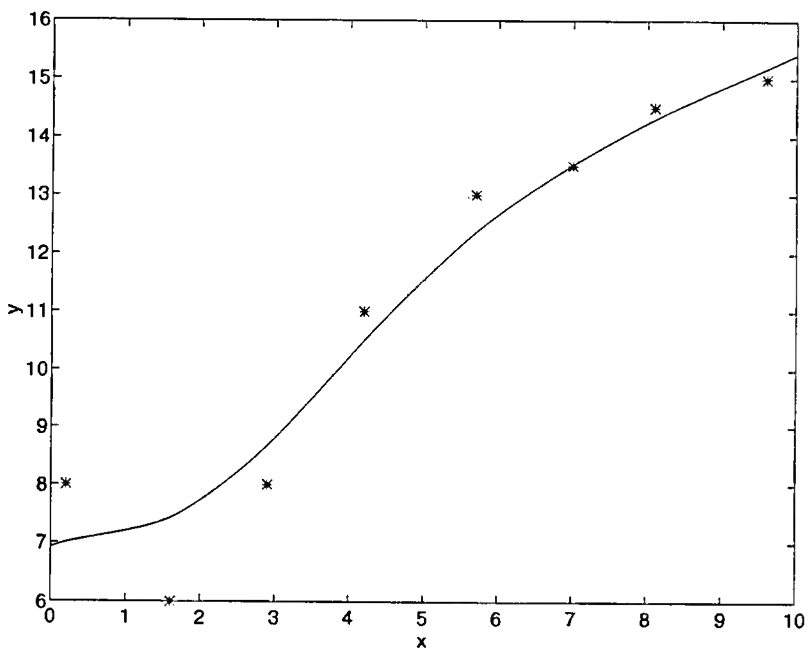

Figure 3: fBm interpolation ($H = 0.9$, $\nu^2 = 1.0$)

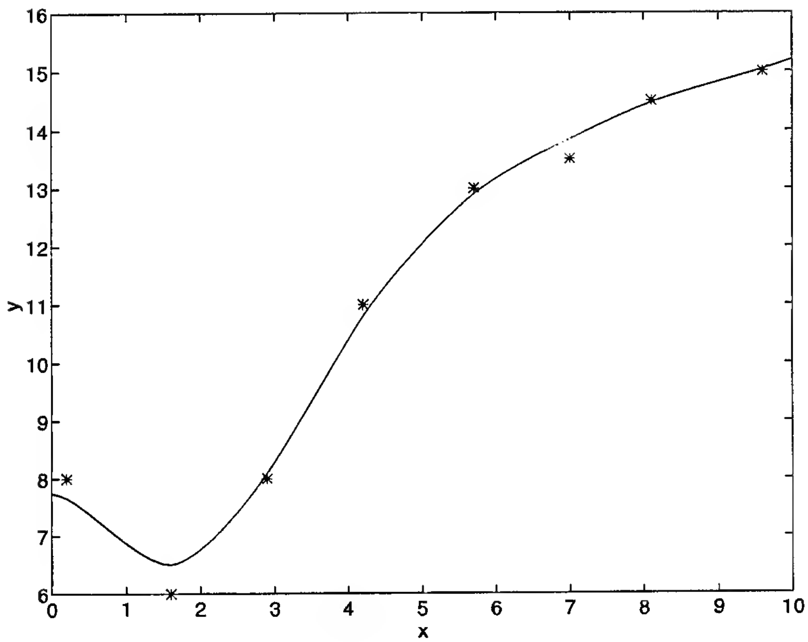

Figure 4: fBm interpolation ($H = 0.9$, ν^2 varies)

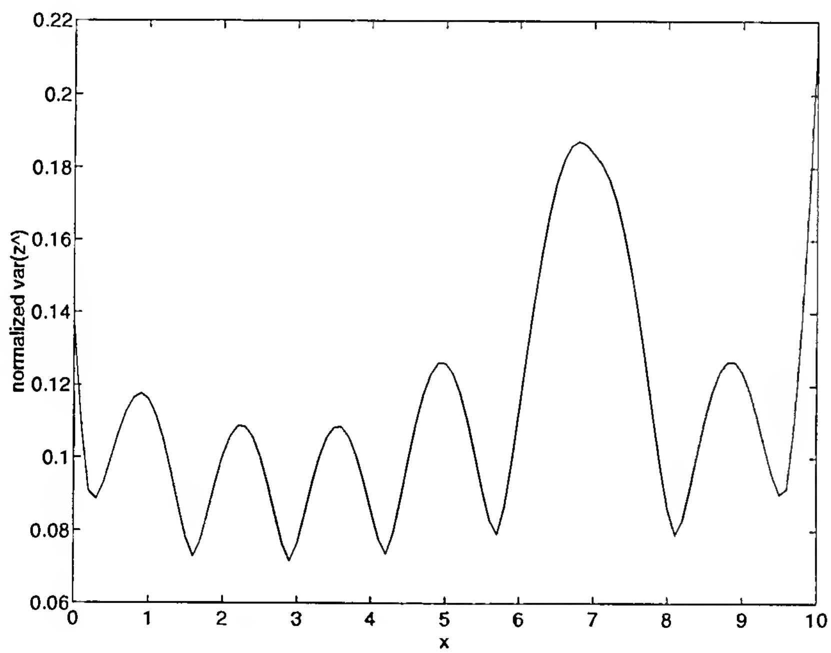

Figure 5: Normalized performance ($\mathbf{var}\{\hat{(z)}\}/\nu^2$) ($H = 0.9$, ν^2 varies)